Time-dependent lift force acting on a particle moving arbitrarily in a pure shear flow, at small Reynolds number

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In this paper we seek an approximate expression for the time-dependent lift force, initially derived by Saffman in the steady case, by using the Asmolov and McLaughlin frequential results. The single-frequency expression derived by these authors can indeed be used to obtain a temporal expression of the lift force acting on a sphere with arbitrary motion, since the problem is linear. The major difficulty lies in the fact that their frequential results are given under a very complicated algebraic expression. Therefore, no analytical inverse Fourier transform can be carried out and the expression of the time-dependent lift force acting on a particle cannot be obtained in the general case. In the present paper, however, it is shown that a closed temporal expression for this force can be found, provided one makes additional approximations. This expression takes the form of a convolution product involving an empirical kernel as well as the slip velocity of the particle.

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Nowadays, it is well known that, even at very small Reynolds number, a particle moving in a shear flow experiences a lift force, the origin of which is inertia effects (see, for example, the well-known experimental results of Segré and Silberberg [1]). Indeed, in a famous paper (including the corrigendum), Saffman [2] emphasized analytically this phenomenon by investigating the steady motion of a sphere in a shear flow, with the following additional assumptions:

$$1 \gg \left(\frac{a^2 G}{\nu}\right)^{1/2} \gg \frac{a V_s}{\nu}, \qquad (1)$$

where a is the radius of the sphere, G is the shear rate of the unperturbed flow, and V_s is the slip velocity of the particle. Using matched asymptotic expansions, Saffman obtained the expression of the lift force acting on the sphere.

More recently, some authors generalized Saffman's result to more complex cases. For instance, McLaughlin [3] dealt with the same problem, by removing the restriction appearing in the second part of the inequality (1). Also, Legendre and Magnaudet [4] obtained the expression of the lift force acting on a fluid inclusion instead of a solid particle. However, one may note that, in general, results concerning the expression of the lift force are obtained by assuming that the perturbed fluid motion equations are steady.

In many natural or industrial flows however, the fluid motion generated by a particle moving in a shear flow may be unsteady. This occurs, for example, when particles collide. In such cases, the difficulty to obtain a valid expression for the lift force increases significantly since both unsteadiness and inertia effects do contribute to the particle lift coefficient, in a very complex and nonadditive manner (as shown numerically by Wakaba and Balachandar [5], for finite Reynolds numbers, and analytically by Candelier and Angilella [6] in the case of a low Reynolds number particle in a rotating flow). Thus, in order to evaluate the lift force acting on the particle in a time-dependent situation, Miyazaki *et al.* [7], and more recently Asmolov and McLaughlin (AM) [8], reconsidered the problem, assuming that the slip velocity of the particle took the form of a time-dependent harmonic function. The interest of this approach lies in the fact that since the perturbed fluid motion equations are linear, these frequential results can be used to derive a temporal expression of the lift force acting on a sphere with arbitrary slip velocity. In this study, we are particularly interested in this last reference since our analysis is based on their results. For this reason, their analysis is described in details in the following.

In order to generalize Saffman's result, AM (see Fig. 1) considered the case of a solid sphere moving with the following slip velocity: $\vec{V}_s(t)=\frac{d\vec{X}(t)}{dt}-\vec{V}^0=U_\omega \exp(-i\omega t)\vec{e}_3$, where $\vec{X}$ is the position of the particle center, given by $\vec{X}(t)=X\vec{e}_1+Z(t)\vec{e}_3$, and $\vec{V}^0=Gx\vec{e}_3$ is the fluid velocity in the absence of the particle (G is the shear rate of the unperturbed flow). Note that their analytical investigations are led under the same assumptions as Saffman [2], that is, Eq. (1).

According to their results [Eq. (24) of Ref. [8], writing $I(\Omega)=2J(\Omega)$], the lift force acting on the sphere in such a situation is given by $F_\omega \exp(-i\omega t)$, where

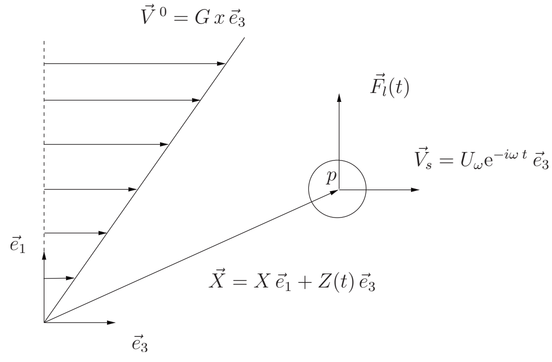

FIG. 1. Configuration of the study considered.

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$$F_\omega = \frac{9}{\pi}\rho a^2 (G\nu)^{1/2} U_\omega J(\Omega), \tag{2}$$

and $\Omega=\omega/G$ is the nondimensional pulsation. $J(\Omega)$ is a complex function which satisfies

$$J(\Omega) \to 2.254 + 3.894\Omega i \quad \text{for} \quad \Omega \ll 1 \tag{3}$$

and

$$J(\Omega) \to \frac{7\pi^2(1+i)}{60\sqrt{2\Omega}} \quad \text{for} \quad \Omega \gg 1. \tag{4}$$

One can readily check that Saffman's result is recovered in the steady case limit $\Omega\to 0$ [Eq. (3)], as expected.

For intermediate angular frequencies Ω, AM have also performed numerical computations of $J(\Omega)$ and their results are given using interpolating functions,

$$\mathrm{Re}[J(\Omega)] = \frac{a_0 + a_1\Omega^1 + a_2\Omega^2 + a_3\Omega^3 + a_4\Omega^{7/2}}{1 + a_5\Omega + a_6\Omega^2 + a_7\Omega^3 + a_8\Omega^4}, \tag{5}$$

$$\mathrm{Im}[J(\Omega)] = \frac{b_1\Omega^1 + b_2\Omega^2 + b_3\Omega^3 + b_1\Omega^{7/2}}{1 + b_5\Omega + b_6\Omega^2 + b_7\Omega^3 + b_8\Omega^4}, \tag{6}$$

where $a_0=2.254$, $a_1=4.528$, $a_2=-2.378$, $a_3=-0.648$, $a_4=2.079$, $a_5=2.009$, $a_6=4.048$, $a_7=-3.545$, $a_8=2.554$, and where $b_1=3.378$, $b_2=1.391$, $b_3=-0.575$, $b_4=1.139$, $b_5=0.523$, $b_6=5.199$, $b_7=-1.396$, $b_8=1.399$.

Let us now introduce the following Fourier transform $F_\omega=\int_{\mathbb{R}} F(t)\exp(i\omega t)dt$. If F_ω corresponds to the lift force when the particle velocity is given by $U_\omega \exp(-i\omega t)\vec{e}_3$, then the force acting on a particle moving with an arbitrary velocity (see, for example, Landau and Lifchitz [9]) is given by

$$F_l(t) = \frac{1}{2\pi}\int_{\mathbb{R}} F_\omega \exp(-i\omega t)d\omega = \mathcal{F}^{-1}(F_\omega), \tag{7}$$

which is nothing but the inverse Fourier transform of F_ω. In particular, in the present study, F_ω is given by (2). However, and before performing any calculations, we need to clarify several important points.

First, the lift force obtained by AM is valid only in the limit where $\Omega \ll \nu/(a^2G)$ and this means that F_ω is unknown, rigorously speaking, when Ω tends toward infinity. However, this difficulty can be overpassed. Indeed, in the present study, assumption (1) requires that $\nu/(a^2G)\gg 1$, so that Ω can be much greater than unity even if $\Omega \ll \nu/(a^2G)$. In such a case, the authors have shown that the function $J(\Omega)$, which is then given by Eq. (4), tends to 0 as Ω increases. In addition, if one considers the problem of a sphere moving with a very high pulsation Ω in a shear flow, and evaluates the order of the various terms appearing into the perturbed fluid motion equations, one can show that if the unsteady term dominates all the convective terms, these equations degenerate into the classical unsteady creeping flow equations. In the case $\Omega \gg 1$, the particle does not experience any lift force, so that we recover the fact that $J(\Omega)$ is null when Ω tends toward infinity. Therefore, in the following, it will be assumed that Eq. (4) is valid for all $\Omega\gg 1$, without any restriction. Nevertheless, the temporal result that will be obtained in the following for the lift force may not be rigorously valid for short times [i.e., for $t=O(a^2/\nu)$]. Note also that in the following, "short times" correspond to $a^2/\nu \ll t \ll 1/G$. Let us now emphasize another important point. We have just seen that in order to obtain the time-dependent lift force, one must calculate the inverse Fourier transform of F_ω which is given by Eq. (7). Unfortunately, and to our knowledge, this inverse Fourier transform cannot be obtained analytically, using the expressions given by (5) and (6). For this reason, we will have to proceed in a different manner in the following: We will seek a temporal expression for the lift force involving an analytical Fourier transform which will be compared to the AM results.

Thus, we first analyze the temporal behavior of the lift force at short times, by using the analytical result (4), since the temporal expression of the lift force should correspond to the inverse Fourier transform of F_ω when $\Omega\gg 1$. Then we should have

$$F_l(t) \sim A_1\mathcal{F}^{-1}(U_\omega) * \mathcal{F}^{-1}\left(\frac{1+i}{\sqrt{\omega}}\right) \quad \text{for} \quad t\to 0,$$

where the symbol $*$ stands for the classical convolution product and the coefficient $A_1=(21\pi/20)(\nu/2)^{1/2}\rho a^2 G$ has been introduced for the sake of simplicity. Since $\mathcal{F}^{-1}[(1+i)/\sqrt{\omega}] =\sqrt{2}/\sqrt{\pi t}$ (see, for example, Landau and Lifchitz [9]), we are led to

$$F_l(t) \sim A_1\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^t \frac{V_s(\tau)}{\sqrt{t-\tau}}d\tau \quad \text{for} \quad t\to 0. \tag{8}$$

This result suggests that if the time-dependent expression of the lift force can be written under the form $F_l(t) =A_1(\sqrt{2}/\sqrt{\pi})\int_0^t K_l(t-\tau)V_s(\tau)d\tau \;\; \forall \;\; t$, then we should have

$$K_l(t) \to \frac{1}{\sqrt{t}} \quad \text{for} \quad t\to 0. \tag{9}$$

Let us now consider a particle, the slip velocity of which experiences an abrupt change, idealized by the classical Heaviside function [i.e., $V_s(t)=V_sH(t)$]. Since the sought expression for the lift force must agree with Saffman's result, we should also have

$$\lim_{t\to\infty} A_1\frac{\sqrt{2}V_s}{\sqrt{\pi}}\int_0^t K_l(t-\tau)d\tau = \frac{9}{\pi}\rho a^2(G\nu)^{1/2}2.254V_s.$$

Once again, we introduce the coefficient $A_2=\frac{9}{\pi}\rho a^2 \times(G\nu)^{1/2}2.254$, in order to simplify this latter expression. The kernel $K_l(t)$ must satisfy the condition

$$\int_0^\infty K_l(\tau)d\tau = \frac{A_2\sqrt{\pi}}{A_1\sqrt{2}} = \text{const.} \tag{10}$$

Clearly, many functions fulfill conditions (9) and (10), like functions of the form $f(\tau)/\sqrt{\tau}$, with $f(0)=1$, provided $f(\tau)$ decreases faster than $1/\sqrt{\tau}$ when τ increases. We now focus on functions of the form

$$K_l(\tau) = \frac{\exp(-\alpha_n \tau^n)}{\sqrt{\tau}} \quad \text{with } n \geq 1, \tag{11}$$

where α_n is a constant, since these functions also admit analytical Fourier transforms. Obviously, condition (9) is satisfied. Also, since

$$\int_0^\infty \frac{\exp(-\alpha_n \tau^n)}{\sqrt{\tau}} d\tau = \frac{\Gamma\left(\frac{1}{2n}\right)\alpha_n^{-1/2n}}{n},$$

where $\Gamma(\cdots)$ is the classical Γ function, one can show that (10) is satisfied by setting the constant α_n to

$$\alpha_n = \{A_1\sqrt{2}\Gamma[1/(2n)]/nA_2\sqrt{\pi}\}^{2n}. \tag{12}$$

Using such a kernel enables one to obtain an analytical Fourier transform of the lift force, and this leads to

$$F_\omega = A_1\sqrt{2/\pi}\mathcal{F}(K_l(t))U_\omega \tag{13}$$

with

$$\mathcal{F}(K_l(t)) = \left(\frac{\alpha_n^{-1/2n}}{n}\right)\left(\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\frac{1+2k}{n}\right)\alpha_n^{(-k/n)}(i\Omega)^k}{k!}\right).$$

Note that if Eq. (13) is rewritten in the form $F_\omega = (9/\pi)\rho a^2(G\nu)^{1/2}U_\omega J_0(\Omega)$, one can readily find that

$$J_0(\Omega) = (7\pi^{3/2}/60)\sqrt{G}\mathcal{F}(K_l(t)) \tag{14}$$

and therefore, the comparison between Eqs. (13) and (2) is straightforward, whatever n. Then, if the real (respectively, the imaginary) part of this function is compared with the interpolated function (5) [respectively, Eq. (6)], for various values of n, the similarity between the curves obtained is striking. If, in addition, we seek the value of n that leads to the best fitting (using a least-squares method), we find approximately $n=3/2$ (see Candelier [10]).

Figure 2 shows the real part (respectively, the imaginary part) of $J_0(\Omega)$ together with Eq. (5) [respectively, Eq. (6)]. Clearly, these curves are very close and this suggests that the time-dependent Saffman's lift force is well approximated by the following expression:

$$F_l = A_1\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^t \frac{\exp[-\alpha_{3/2}(t-\tau)^{3/2}]}{\sqrt{t-\tau}}V_s(\tau)d\tau \tag{15}$$

which, by construction, sticks to the expected lift force at short times, and tends to the classical Saffman lift force in the steady limit.

In order to test expression (15), let us now focus on the case of a particle experiencing an abrupt change of slip velocity [i.e., $V_s(t)=V_sH(t)$]. In such conditions, the lift force (15) reads as

$$\frac{F_l(t)}{F_{\text{Saffman}}} = \frac{A_1}{A_2}\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^t \frac{\exp[-\alpha_{3/2}(t-\tau)^{3/2}]}{\sqrt{t-\tau}}d\tau. \tag{16}$$

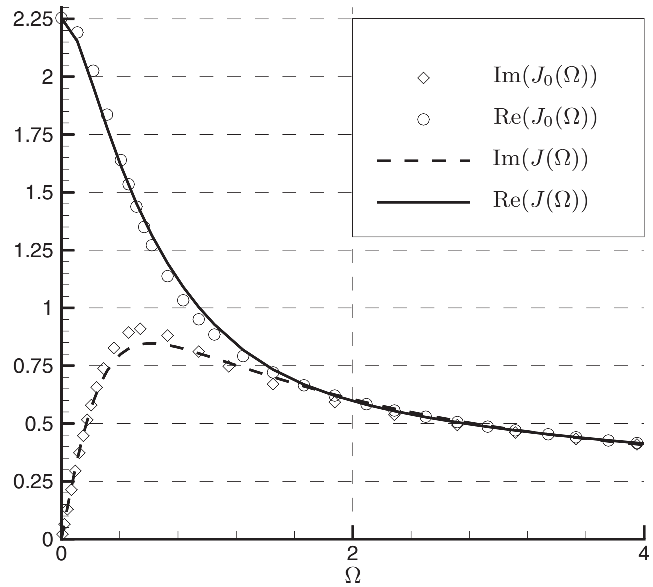

FIG. 2. Plot of Re[$J(\Omega)$] [Eq. (5)] and Re[$J_0(\Omega)$] [Eq. (14)] versus Ω and of Im[$J(\Omega)$] [Eq. (6)] and Im[$J_0(\Omega)$] versus Ω [Eq. (14)].

Figure 3 shows the lift [numerical solution of (16)] in response to such an abrupt change, together with the corresponding results of AM. Note that the AM results have also been obtained numerically, but for a T-periodic slip velocity with $T \gg 1/G$, $V_s(t)=V_s[1/2+2\Sigma_{n=1}^{\infty}\sin(A_n t/T)/A_n]$, where $A_n=\pi(2n-1)$, in order to make use of Fourier series instead of calculating the inverse Fourier transform given by Eq. (7). Writing the slip velocity under this form enables one to carry out more easily the force corresponding to AM results which reads as

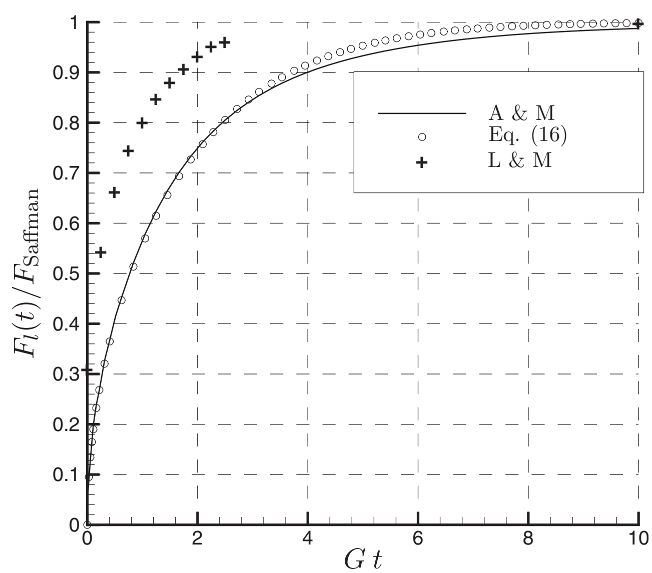

FIG. 3. Temporal evolution of the lift force in response to an abrupt change of the slip velocity, corresponding to the AM results [Eq. (17)], to Eq. (16), and to the numerical results of Legendre and Magnaudet (denoted LM) [4].

$$\frac{F_l(t)}{F_{\text{Saffman}}} = \frac{1}{2} - \frac{2}{2.254}\left(\sum_{n=1}^{\infty}\frac{\text{Im}[J(A_n/T)]\cos(A_n t/T)}{A_n}\right) + \frac{2}{2.254}\left(\sum_{n=1}^{\infty}\frac{\text{Re}[J(A_n/T)]\sin(A_n t/T)}{A_n}\right). \quad (17)$$

The two responses are in very good agreement, and one can see that the time taken by this force to reach its steady value is of the order of $O(G^{-1})$. Note that this result is physically sound, since this time also corresponds to the time required for vorticity (created by the change of slip velocity) to diffuse to the distance $\sqrt{\nu/G}$ which corresponds to the distance where the shear-based convective terms balance the viscous term in the perturbed fluid motion equations. Note that this lift response can be also qualitatively compared to the numerical results of Legendre and Magnaudet [4], see again Fig. 3, taking care of the fact that their situation is not rigorously identical to the one described in this paper, since they have $aV_s/\nu = (a^2G/\nu)^{1/2} = 0.25$ [the second part of (1) is not satisfied]. Also, the particle considered by these authors is a fluid inclusion instead of a solid sphere.

Choosing $n=3/2$ for the kernel (11) leads to a relaxation of the form $\exp(-t^{3/2})/\sqrt{t}$ where the coefficient 3/2 may be explained by the following arguments: since $\alpha_{3/2}$ is proportional to $G^{3/2}$ (the term $\alpha_{3/2}t^{3/2}$ is dimensionless), one can write $\alpha_{3/2}t^{3/2} \propto Gt\sqrt{\nu t}/(\sqrt{\nu/G})$. This last expression is the ratio between two length scales. We are very familiar with the denominator length scale, since it corresponds to the distance where the shear-based convective terms become significant compared to the viscous terms into the perturbed fluid motion equations. The numerator is a characteristic length scale resulting from the combined effect of diffusion $(\sqrt{\nu t})$ and advection by a shear flow (see, for example, Rhines and Young [11]). Therefore, it seems natural to see such a time scaling appear in this empirical kernel.

Note also that the kernel of the temporal lift force obtained at small Reynolds number is monotonic, in contrast with at finite Reynolds number kernels (see the numerical results of Wakaba and Balachandar [5]). Actually, it has been shown (see Legendre and Magnaudet [4], or again Wakaba and Balachandar [5]) that the physical mechanisms governing the lift force are very different at low Reynolds number and at finite Reynolds number. This probably explains the noted differences between the kernels of the lift force. However, this point requires further investigations, and therefore provides an interesting prospect of this study. Finally, note that if one needs to obtain a more accurate expression for the temporal lift force, one can check that the difference between the two functions J and J_0 can be very well fitted with Gaussian functions of the form $J_i = C_m i^m(\omega/G)^m \times \exp[-C_{m+1}(\omega/G)^2]$, which also admit real analytical inverse Fourier transforms (see Candelier [10]).

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[1] G. Segré and A. Silberberg, J. Fluid Mech. **14**, 136 (1962).
[2] P. G. Saffman, J. Fluid Mech. **22**, 385 (1965).
[3] J. B. McLaughlin, J. Fluid Mech. **224**, 261 (1991).
[4] D. Legendre and J. Magnaudet, J. Fluid Mech. **368**, 81 (1998).
[5] L. Wakaba and S. Balachandar, Int. J. Multiphase Flow **31**, 996 (2005).
[6] F. Candelier and J. R. Angilella, Phys. Rev. E **73**, 047301 (2006).
[7] K. Miyazaki, D. Bedeaux, and J. Bonet Avalos, J. Fluid Mech. **296**, 373 (1995).
[8] E. Asmolov and J. B. McLaughlin, Int. J. Multiphase Flow **183**, 199 (1999).
[9] L. Landau and E. Lifchitz, *Mécanique des Fluides*, Vol. 6 of Physique théorique (Edition Mir, Paris, 1989).
[10] F. Candelier, Ph.D. thesis, INPL, 2005.
[11] P. B. Rhines and W. R. Young, J. Fluid Mech. **133**, 133 (1983).